Goldstein 9.5

$$\begin{cases} Q \equiv \arctan\left(\frac{\alpha q}{p}\right) \\ P \equiv \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right) \end{cases}$$

$$\Rightarrow \quad q = \frac{P}{\alpha}\frac{\sin Q}{\cos Q}$$

it's then tempting to use the third generating function relation to solve for F_3:

$$\begin{cases} q_i = -\frac{\partial F_3}{\partial p_i} \\ p_i = -\frac{\partial F_3}{\partial Q_i} \end{cases}$$

This suggests $F_3 = -\frac{p^2}{2\alpha}\frac{\sin Q}{\cos Q}$, we need to check whether it gives the correct form of P:

$$-\frac{\partial F_3}{\partial Q} = \frac{p^2}{2\alpha}\left[1 + \frac{\sin^2 Q}{\cos^2 Q}\right]$$

from $q = \frac{P}{\alpha}\frac{\sin Q}{\cos Q}$, we have $\frac{\sin^2 Q}{\cos^2 Q} = \frac{\alpha^2 q^2}{P^2}$ and it gives the right form of P.

Thus the canonical transformation is satisfied with $\boxed{F_3 = -\frac{p^2}{2\alpha}\tan Q}$

relation

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2.7.2024